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Noether's second theorem in a general setting: reducible gauge theories

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Abstract
We prove Noether's direct and inverse second theorems for Lagrangian systems on fibre bundles in the case of gauge symmetries depending on derivatives of dynamic variables and parameters of an arbitrary order. The appropriate notions of a reducible gauge symmetry and Noether identity are formulated, and their equivalence by means of a certain intertwining operator is proved.

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1. Introduction

Different variants of Noether's second theorem state that, if a Lagrangian admits symmetries depending on parameters, its variational derivatives obey certain relations, called the Noether identity. In a rather general setting, this theorem has been formulated in [8]. Gauge symmetries and Noether identities need not be independent, and one speaks of N-stage reducible gauge symmetries and Noether identities. The notion of a reducible Noether identity has come from that of a reducible constraint [7], but it involves differential operators. Note that the conventional Batalin–Vilkovisky (BV) quantization of a classical gauge system necessarily starts with studying an hierarchy of its gauge symmetries and Noether identities in order to define the multiplet of ghosts and antifields, and to construct the so-called gauge-fixed Lagrangian [2, 13]. It should also be emphasized that, if a gauge symmetry is reducible, the components of a Noether current in classical field theory and the Ward identities in quantum field theory fail to be independent.

We present Noether's second theorem and its inverse (theorem 4.2) for Lagrangian systems on a fibre bundle $Y \to X$ in the case of gauge symmetries depending on derivatives of dynamic variables and parameters of an arbitrary order. Bearing in mind the extension of the BV quantization scheme to an arbitrary base manifold [3, 12], we pay particular attention to global aspects of Noether's second theorem. For this purpose, we consider a Lagrangian formalism

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on the composite fibre bundle $E \to Y \to X$, where $E \to Y$ is a vector bundle of gauge parameters. Accordingly, a gauge symmetry is represented by a linear differential operator υ on E taking its values in the vertical tangent bundle VY of $Y \to X$.

The Noether identity for a Lagrangian L is defined by a differential operator Δ on the fibre bundle (2.12) which takes its values in the density-dual

$$\overline{E}^* = E^* \underset{Y}{\otimes} \overset{n}{\wedge} T^*X \tag{1.1}$$

of $E \to Y$ and whose kernel contains the image of the Euler–Lagrange operator δL of L, i.e.,

$$\Delta \circ \delta L = 0. \tag{1.2}$$

Expressed in these terms, Noether's second theorem and its inverse follow at once from the first variational formula (proposition 3.1) and the properties of differential operators on dual fibre bundles (theorem 8.1). Namely, there exists the intertwining operator $\eta(\upsilon) = \Delta$ (A.4), $\eta(\Delta) = \upsilon$ (A.5) such that

$$\eta(\eta(\upsilon)) = \upsilon, \qquad \eta(\eta(\Delta)) = \Delta, \tag{1.3}$$

$$\eta(\upsilon \circ \upsilon') = \eta(\upsilon') \circ \eta(\upsilon), \qquad \eta(\Delta' \circ \Delta) = \eta(\Delta) \circ \eta(\Delta'). \tag{1.4}$$

The appropriate notions of a reducible Noether identity and gauge symmetry are formulated, and their equivalence with respect to the intertwining operator η is proved (theorem 5.3).

The following two examples aim to illustrate our exposition: (i) the gauge theory of principal connections for which gauge transformations need not be vertical, e.g., the topological gauge theory with the global Chern–Simons Lagrangian and the Yang–Mills gauge theory with a dynamic metric field, (ii) a gauge system of skew symmetric tensor fields with a reducible gauge symmetry, e.g., the topological BF theory.

2. Lagrangian formalism on fibre bundles

The Lagrangian formalism on a fibre bundle $Y \to X$ is phrased in terms of the following graded differential algebra (henceforth GDA) [1, 10, 12, 15]. The finite-order jet manifolds of $Y \to X$ form an inverse system

$$X \overset{\pi}{\longleftarrow} Y \overset{\pi^1_0}{\longleftarrow} J^1Y \longleftarrow \cdots J^{r-1}Y \overset{\pi^r_{r-1}}{\longleftarrow} J^rY \longleftarrow \cdots. \tag{2.1}$$

In the following, the index $r = 0$ stands for Y. Accordingly, we have the direct system

$$\mathcal{O}^*X \overset{\pi^*}{\longrightarrow} \mathcal{O}^*Y \overset{\pi^{1*}_0}{\longrightarrow} \mathcal{O}^*_1Y \longrightarrow \cdots \mathcal{O}^*_{r-1}Y \overset{\pi^r_{r-1}{}^*}{\longrightarrow} \mathcal{O}^*_rY \longrightarrow \cdots \tag{2.2}$$

of GDAs $\mathcal{O}^*_rY$ of exterior forms on jet manifolds J^rY with respect to the pull-back monomorphisms $\pi^r_{r-1}{}^*$. Its direct limit $\mathcal{O}^*_\infty[Y]$ is a GDA consisting of all exterior forms on finite-order jet manifolds modulo the pull-back identification.

The projective limit $\left(J^\infty Y, \pi^\infty_r : J^\infty Y \to J^rY\right)$ of the inverse system (2.1) is a Fréchet manifold. A bundle atlas $\{(U_Y; x^\lambda, y^i)\}$ of $Y \to X$ yields the coordinate atlas

$$\left\{\left(\left(\pi^\infty_0\right)^{-1}(U_Y); x^\lambda, y^i_\Lambda\right)\right\}, \qquad y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \qquad 0 \leqslant |\Lambda|, \tag{2.3}$$

of $J^\infty Y$, where $\Lambda = (\lambda_k \cdots \lambda_1)$ is a symmetric multi-index, $\lambda + \Lambda = (\lambda\lambda_k \cdots \lambda_1)$, and

$$d_\lambda = \partial_\lambda + \sum_{0\leqslant|\Lambda|} y^i_{\lambda+\Lambda}\partial^\Lambda_i, \qquad d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1} \tag{2.4}$$

are total derivatives. There is the GDA epimorphism $\mathcal{O}^*_\infty[Y] \to \mathcal{O}^*_\infty[U_Y]$ obtained as the restriction of $\mathcal{O}^*_\infty Y$ to chart (2.3). Then $\mathcal{O}^*_\infty[Y]$ can be written in a coordinate form where the horizontal 1-forms $\{dx^\lambda\}$ and the contact 1-forms $\{\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}\,dx^\lambda\}$ are generating elements of the $\mathcal{O}^0_\infty[U_Y]$-algebra $\mathcal{O}^*_\infty[U_Y]$. Though $J^\infty Y$ is not a smooth manifold, the coordinate transformations of elements of $\mathcal{O}^*_\infty[Y]$ are smooth since they are exterior forms on finite-order jet manifolds.

There is the canonical decomposition $\mathcal{O}^*_\infty[Y] = \oplus\mathcal{O}^{k,m}_\infty[Y]$ of $\mathcal{O}^*_\infty[Y]$ into $\mathcal{O}^0_\infty[Y]$-modules $\mathcal{O}^{k,m}_\infty[Y]$ of k-contact and m-horizontal forms together with the corresponding projectors

$$h_k : \mathcal{O}^*_\infty[Y] \to \mathcal{O}^{k,*}_\infty[Y], \qquad h^m : \mathcal{O}^*_\infty[Y] \to \mathcal{O}^{*,m}_\infty[Y].$$

Accordingly, the exterior differential on $\mathcal{O}^*_\infty[Y]$ is split into the sum $d = d_H + d_V$ of the nilpotent total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \qquad d_V(\phi) = \theta^i_\Lambda \wedge \partial^\Lambda_i \phi, \qquad \phi \in \mathcal{O}^*_\infty[Y].$$

In particular, any finite-order Lagrangian on a fibre bundle $Y \to X$ is a density

$$L = \mathcal{L}\omega \in \mathcal{O}^{0,n}_\infty[Y], \qquad \omega = dx^1 \wedge \cdots \wedge dx^n, \qquad n = \dim X. \tag{2.5}$$

In the framework of Lagrangian formalism, we deal with differential operators of the following type. Let

$$W \to Y \to X, \qquad Z \to Y \to X$$

be composite bundles, including $W = Y$, and let $Z \to Y$ be a vector bundle. By a kth-order differential operator on $W \to X$ taking its values in $Z \to X$ is throughout meant a bundle morphism

$$\Delta : J^k W \underset{Y}{\longrightarrow} Z. \tag{2.6}$$

Its kernel Ker Δ is defined as the inverse image of the canonical zero section of $Z \to Y$. In an equivalent way, the differential operator (2.6) is represented by a section Δ of the vector bundle $J^k W \underset{Y}{\times} Z \to J^k W$. Given bundle coordinates (x^λ, y^i, w^r) on W and (x^λ, y^i, z^A) on Z with respect to the fibre basis $\{e_A\}$ for $Z \to Y$, this section reads

$$\Delta = \Delta^A\big(x^\lambda, y^i_\Lambda, w^r_\Lambda\big)e_A, \qquad 0 \leqslant |\Lambda| \leqslant k. \tag{2.7}$$

Then the differential operator (2.6) is also represented by an element

$$\Delta = \Delta^A\big(x^\lambda, y^i_\Lambda, w^r_\Lambda\big)z_A \in \mathcal{O}^0_\infty[W \times Z^*] \tag{2.8}$$

of the GDA $\mathcal{O}^*_\infty[W \underset{X}{\times} Z^*]$, where $Z^* \to Y$ is the dual of $Z \to Y$ with coordinates (x^λ, y^i, z_A).

If $W \to Y$ is a vector bundle, a differential operator Δ (2.6) on the composite bundle $W \to Y \to X$ is said to be linear if it is linear on the fibres of the vector bundle $J^k W \to J^k Y$. In this case, its representations (2.7) and (2.8) take the form

$$\Delta = \sum_{0\leqslant|\Xi|\leqslant k} \Delta^{A,\Xi}_r\big(x^\lambda, y^i_\Lambda\big)w^r_\Xi e_A, \qquad 0 \leqslant |\Lambda| \leqslant k, \tag{2.9}$$

$$\Delta = \sum_{0\leqslant|\Xi|\leqslant k} \Delta^{A,\Xi}_r\big(x^\lambda, y^i_\Lambda\big)w^r_\Xi z_A, \qquad 0 \leqslant |\Lambda| \leqslant k. \tag{2.10}$$

In particular, every Lagrangian L (2.5) defines the Euler–Lagrange operator

$$\delta L = \sum_{0\leqslant|\Lambda|} (-1)^{|\Lambda|} d_\Lambda\big(\partial^\Lambda_i \mathcal{L}\big)\, dy^i \wedge \omega \tag{2.11}$$

on Y taking the values in the vector bundle

$$V^*Y \underset{Y}{\otimes} \overset{n}{\underset{X}{\wedge}} T^*X \to Y. \tag{2.12}$$

It is represented by the exterior form

$$\delta L = \mathcal{E}_i \theta^i \wedge \omega = \sum_{0\leqslant|\Lambda|} (-1)^{|\Lambda|} d_\Lambda\big(\partial_i^\Lambda \mathcal{L}\big)\theta^i \wedge \omega \in \mathcal{O}_\infty^{1,n}[Y], \tag{2.13}$$

where

$$\delta\phi = \sum_{0\leqslant|\Lambda|} (-1)^{|\Lambda|}\theta^i \wedge \big[d_\Lambda\big(\partial_i^\Lambda \rfloor \mathrm{d}\phi\big)\big], \qquad \phi \in \mathcal{O}_\infty^{*,n}[Y], \tag{2.14}$$

is the variational operator acting on $\mathcal{O}_\infty^{*,n}[Y]$ so that $\delta \circ d_H = 0$ and $\delta \circ \delta = 0$. There is the canonical decomposition

$$\mathrm{d}L = \delta L - d_H \Xi, \tag{2.15}$$

where $\Xi_L = L + \Xi$ is a Lepagean equivalent of L. It reads

$$\Xi_L = L + \sum_{s=0} F_i^{\lambda\nu_s\cdots\nu_1}\theta^i_{\nu_s\cdots\nu_1} \wedge \omega_\lambda,$$

$$F_i^{\nu_k\cdots\nu_1} = \partial_i^{\nu_k\cdots\nu_1}\mathcal{L} - d_\lambda F_i^{\lambda\nu_k\cdots\nu_1} + h_i^{\nu_k\cdots\nu_1}, \qquad \omega_\lambda = \partial_\lambda \rfloor \omega,$$

where functions h obey the relations $h_i^\nu = 0$, $h_i^{(\nu_k\nu_{k-1})\cdots\nu_1} = 0$ [14].

Remark 2.1. Given a Lagrangian L and its Euler–Lagrange operator δL (2.13), we further abbreviate $A \approx 0$ with an equality which holds on-shell. This means that A is an element of a module over the ideal I_L of the ring $\mathcal{O}_\infty^0[Y]$ which is locally generated by the variational derivatives $\mathcal{E}_i$ and their total derivatives $d_\Lambda \mathcal{E}_i$. One says that I_L is a differential ideal because, if a local function f belongs to I_L, then every total derivative $d_\Lambda f$ does as well.

Remark 2.2. We will use the relations

$$\sum_{0\leqslant|\Lambda|\leqslant k} B^\Lambda d_\Lambda A' = \sum_{0\leqslant|\Lambda|\leqslant k} (-1)^{|\Lambda|} d_\Lambda(B^\Lambda)A' + d_H\sigma, \tag{2.16}$$

$$\sum_{0\leqslant|\Lambda|\leqslant k} (-1)^{|\Lambda|} d_\Lambda(B^\Lambda A) = \sum_{0\leqslant|\Lambda|\leqslant k} \eta(B)^\Lambda d_\Lambda A, \tag{2.17}$$

$$\eta(B)^\Lambda = \sum_{0\leqslant|\Sigma|\leqslant k-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma B^{\Sigma+\Lambda}, \qquad C^a_b = \frac{b!}{a!(b-a)!}, \tag{2.18}$$

for arbitrary exterior forms $A' \in \mathcal{O}_\infty^{*,n}[Q]$, $A \in \mathcal{O}_\infty^{*}[Q]$ and local functions $B^\Lambda \in \mathcal{O}_\infty^0[Q]$ on jet manifolds of a fibre bundle $Q \to X$. Since $\sum_{a=0}^k (-1)^a C^a_k = 0$ for $k > 0$, it is easily verified that

$$(\eta \circ \eta)(B)^\Lambda = B^\Lambda. \tag{2.19}$$

3. Gauge symmetries in a general setting

Let $\mathfrak{d}\mathcal{O}_\infty^0 Y$ be the $\mathcal{O}_\infty^0[Y]$-module of derivations of the $\mathbb{R}$-algebra $\mathcal{O}_\infty^0[Y]$. Any $\vartheta \in \mathfrak{d}\mathcal{O}_\infty^0 Y$ yields a graded derivation (the interior product) $\vartheta \rfloor \phi$ of the GDA $\mathcal{O}_\infty^*[Y]$ given by the relations

$$\vartheta\rfloor \mathrm{d}f = \vartheta(f), \qquad f \in \mathcal{O}^0_\infty[Y],$$
$$\vartheta\rfloor(\phi\wedge\sigma) = (\vartheta\rfloor\phi)\wedge\sigma + (-1)^{|\phi|}\phi\wedge(\vartheta\rfloor\sigma), \qquad \phi,\sigma\in\mathcal{O}^*_\infty[Y],$$

and a derivation $\mathbf{L}_\vartheta$ (the Lie derivative) which satisfies the conditions

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor\mathrm{d}\phi + \mathrm{d}(\vartheta\rfloor\phi), \qquad \phi\in\mathcal{O}^*_\infty[Y], \tag{3.1}$$
$$\mathbf{L}_\vartheta(\phi\wedge\phi') = \mathbf{L}_\vartheta(\phi)\wedge\phi' + \phi\wedge\mathbf{L}_\vartheta(\phi'), \tag{3.2}$$
$$\mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi). \tag{3.3}$$

Relative to an atlas (2.3), a derivation $\vartheta\in\mathfrak{d}\mathcal{O}^0_\infty Y$ reads

$$\vartheta = \vartheta^\lambda\partial_\lambda + \vartheta^i\partial_i + \sum_{|\Lambda|>0}\vartheta^i_\Lambda\partial^\Lambda_i, \tag{3.4}$$

where the tuple of derivations $\{\partial_\lambda, \partial^\Lambda_i\}$ is defined as the dual of the set $\{\mathrm{d}x^\lambda, \mathrm{d}y^i_\Lambda\}$ of generating elements for the $\mathcal{O}^0_\infty[Y]$-algebra $\mathcal{O}^*_\infty[Y]$ with respect to the interior product $\rfloor$ [12].

A derivation ϑ (3.4) is called contact if the Lie derivative $\mathbf{L}_\vartheta$ (3.1) preserves the contact ideal of the GDA $\mathcal{O}^*_\infty[Y]$ generated by contact forms. A derivation ϑ (3.4) is contact iff

$$\vartheta^i_\Lambda = d_\Lambda\left(\vartheta^i - y^i_\mu\vartheta^\mu\right) + y^i_{\mu+\Lambda}\vartheta^\mu, \qquad 0<|\Lambda|. \tag{3.5}$$

Any contact derivation admits the canonical horizontal splitting

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + \left(\upsilon^i\partial_i + \sum_{0<|\Lambda|} d_\Lambda\upsilon^i\partial^\Lambda_i\right), \qquad \upsilon^i = \vartheta^i - y^i_\mu\vartheta^\mu. \tag{3.6}$$

Its vertical part ϑ_V is completely determined by the first summand

$$\upsilon = \upsilon^i\left(x^\lambda, y^i_\Lambda\right)\partial_i, \qquad 0\leqslant|\Lambda|\leqslant k. \tag{3.7}$$

This is a section of the pull-back $VY\underset{Y}{\times}J^kY\to J^kY$ of the vertical tangent bundle $VY\to Y$ onto J^kY [6] and, thus, it is a kth-order VY-valued differential operator on Y. One calls υ (3.7) a generalized vector field on Y.

Proposition 3.1. *It follows from splitting (2.15) that the Lie derivative of a Lagrangian L (2.5) along a contact derivation ϑ (3.9) fulfils the first variational formula*

$$\mathbf{L}_\vartheta L = \upsilon\rfloor\delta L + d_H(h_0(\vartheta\rfloor\Xi_L)) + \mathcal{L}d_V(\vartheta_H\rfloor\omega), \tag{3.8}$$

where Ξ_L is a Lepagean equivalent of L [12].

A contact derivation ϑ (3.9) is called variational if the Lie derivative (3.8) is d_H-exact, i.e., $\mathbf{L}_\vartheta L = d_H\sigma$, $\sigma\in\mathcal{O}^{0,n-1}_\infty[Y]$. A glance at expression (3.8) shows that: (i) a contact derivation ϑ is variational only if it is projectable onto X, (ii) ϑ is variational iff its vertical part ϑ_V is well, (iii) it is variational if $\upsilon\rfloor\delta L$ is d_H-exact. By virtue of item (ii), we can restrict our consideration to vertical contact derivations

$$\vartheta = \sum_{0\leqslant|\Lambda|} d_\Lambda\upsilon^i\partial^\Lambda_i. \tag{3.9}$$

A generalized vector field υ (3.7) is called a variational symmetry of a Lagrangian L if it generates a variational vertical contact derivation (3.9).

A Lagrangian system on a fibre bundle $Y\to X$ is said to be a gauge theory if its Lagrangian L admits a family of variational symmetries parametrized by elements of a vector bundle $E\to Y$ as follows.

Let $E\to Y$ be a vector bundle coordinated by (x^λ, y^i, ξ^r). Given a Lagrangian L on Y, let us consider its pull-back, say again L, onto E. Let ϑ_E be a contact derivation of the $\mathbb{R}$-algebra $\mathcal{O}^0_\infty[E]$, whose restriction

$$\vartheta=\vartheta_E|_{\mathcal{O}^0_\infty Y}=\sum_{0\leqslant|\Lambda|} d_\Lambda \upsilon^i\partial_i^\Lambda \tag{3.10}$$

to $\mathcal{O}^0_\infty[Y]\subset\mathcal{O}^0_\infty[E]$ is linear in coordinates ξ^r_Ξ. It is determined by a generalized vector field υ_E on E whose canonical projection

$$\upsilon: J^kE\xrightarrow{\upsilon_E} VE\to E\underset{Y}{\times} VY$$

(see the exact sequence (3.12)) is a linear VY-valued differential operator

$$\upsilon=\sum_{0\leqslant|\Xi|\leqslant m}\upsilon^{i,\Xi}_r\left(x^\lambda, y^i_\Sigma\right)\xi^r_\Xi\partial_i \tag{3.11}$$

on $E\to Y\to X$. Let ϑ_E be a variational symmetry of a Lagrangian L on E, i.e.,

$$\mathbf{L}_{\vartheta_E}L=d_H\sigma.$$

Then one says that υ (3.11) is a gauge symmetry of a Lagrangian L.

Remark 3.1. Note that any generalized vector field υ (3.11) gives rise to a generalized vector field υ_E on E and, thus, defines a contact derivation ϑ_E of $\mathcal{O}^0_\infty[E]$. Indeed, let us consider the exact sequence of vector bundles

$$0\to V_YE\to VE\to E\underset{Y}{\times}VY\to 0, \tag{3.12}$$

where V_YE is the vertical tangent bundle of $E\to Y$. Its splitting Γ lifts υ to the generalized vector field $\upsilon_E=\Gamma\circ\upsilon$ on E such that the Lie derivative

$$\mathbf{L}_{\vartheta_E}L=\upsilon\rfloor\delta L+d_H(\vartheta\rfloor\Xi_L) \tag{3.13}$$

depends only on υ, but not a lift Γ.

Remark 3.2. If υ (3.11) is a gauge symmetry, we obtain from the first variational formula (3.13) the weak conservation law

$$0\approx d_H(\vartheta\rfloor\Xi_L-\sigma), \tag{3.14}$$

where

$$J=\vartheta\rfloor\Xi_L=\sum_{0\leqslant|\Lambda|}J^{\lambda,\Lambda}_r\xi^r_\Lambda\omega_\lambda, \tag{3.15}$$

is a Noether current.

4. Noether's second theorem

Let us start with the notion of a Noether identity.

Definition 4.1. *Given a Lagrangian L (2.5) and its Euler–Lagrange operator δL (2.13), let $E\to Y$ be a vector bundle and Δ a linear differential operator of order $0\leqslant m$ on the composite bundle (2.12) with the values in the density-dual $\overline{E}^*$ (1.1) of E which obeys condition (1.2). This condition is called the Noether identity, and Δ is the Noether operator.*

Given bundle coordinates $(x^\lambda, y^i, \overline{y}_i)$ on the fibre bundle (2.12) and (x^λ, y^i, ξ^r) on E, the Noether operator Δ is represented by the density

$$\Delta = \Delta_r \xi^r \omega = \sum_{0\leqslant|\Lambda|\leqslant m} \Delta_r^{i,\Lambda}\big(x^\lambda, y^j_\Sigma\big)\overline{y}_{\Lambda i}\xi^r\omega \in \mathcal{O}^{0,n}_\infty[E \underset{Y}{\times} V^*Y], \qquad 0\leqslant|\Sigma|\leqslant m. \tag{4.1}$$

Then the Noether identity (1.2) takes the coordinate form

$$\sum_{0\leqslant|\Lambda|\leqslant m} \Delta_r^{i,\Lambda} d_\Lambda \mathcal{E}_i \xi^r\omega = 0. \tag{4.2}$$

Theorem 4.2. *If a Lagrangian L (2.5) admits a gauge symmetry υ (3.11), its Euler–Lagrange operator obeys the Noether identity (4.2) where the Noether operator (4.1) is*

$$\Delta = \eta(\upsilon) = \sum_{0\leqslant|\Sigma|\leqslant m} (-1)^{|\Sigma|} d_\Sigma\big(\upsilon_r^{i,\Sigma}\overline{y}_i\big)\xi^r\omega = \sum_{0\leqslant|\Lambda|\leqslant m} \eta(\upsilon)_r^{i,\Lambda}\overline{y}_{\Lambda i}\xi^r\omega,$$

$$\eta(\upsilon)_r^{i,\Lambda} = \sum_{0\leqslant|\Sigma|\leqslant m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma \upsilon_r^{i,\Sigma+\Lambda}. \tag{4.3}$$

Conversely, if the Euler–Lagrange operator of a Lagrangian L obeys the Noether identity (4.2), this Lagrangian admits a gauge symmetry υ (3.11) where

$$\upsilon = \eta(\Delta) = \sum_{0\leqslant|\Sigma|\leqslant m} (-1)^{|\Sigma|} d_\Sigma\big(\Delta_r^{i,\Sigma}\xi^r\big)\partial_i = \sum_{0\leqslant|\Lambda|\leqslant m} \eta(\Delta)_r^{i,\Lambda}\xi^r_\Lambda\partial_i, \tag{4.4}$$

$$\eta(\Delta)_r^{i,\Lambda} = \sum_{0\leqslant|\Sigma|\leqslant m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma \Delta_r^{i,\Sigma+\Lambda}. \tag{4.5}$$

Proof. Given an operator υ (3.11), the operator $\Delta = \eta(\upsilon)$ (4.3) is defined in accordance with theorem 8.1 in the appendix. Since the density

$$\upsilon\rfloor\delta L = \upsilon^i\mathcal{E}_i\omega = \sum_{0\leqslant|\Xi|\leqslant m} \upsilon_r^{i,\Xi}\xi^r_\Xi\mathcal{E}_i\omega$$

is d_H-exact, the Noether identity

$$\delta(\upsilon\rfloor\delta L) = \eta(\upsilon)\circ\delta L = 0$$

holds. Conversely, any operator Δ (4.1) defines the generalized vector field $\upsilon = \eta(\Delta)$ (4.4). Due to the Noether identity (4.2), we obtain

$$0 = \sum_{0\leqslant|\Lambda|\leqslant m} \xi^r\Delta_r^{i,\Lambda}d_\Lambda\mathcal{E}_i\omega = \sum_{0\leqslant|\Lambda|\leqslant m} (-1)^{|\Lambda|} d_\Lambda\big(\xi^r\Delta_r^{i,\Lambda}\big)\mathcal{E}_i\omega + d_H\sigma$$
$$= \sum_{0\leqslant|\Xi|\leqslant m} \upsilon_r^{i,\Xi}\xi^r_\Xi\mathcal{E}_i\omega + d_H\sigma = \upsilon\rfloor\delta L + d_H\sigma,$$

i.e., υ is a gauge symmetry of L. □

By virtue of relations (1.3), there is one-to-one correspondence between gauge symmetries of a Lagrangian L and the Noether identities for δL.

Example 4.1. If a gauge symmetry

$$\upsilon = \left(\upsilon_r^i \xi^r + \upsilon_r^{i,\mu} \xi_\mu^r\right) \partial_i \tag{4.6}$$

is of first jet order in parameters, the corresponding Noether operator (4.3) and Noether identity take the form

$$\Delta = \left[\left(\upsilon_r^i - d_\mu \upsilon_r^{i,\mu}\right) \overline{y}_i - \upsilon_r^{i,\mu} \overline{y}_{\mu i}\right] \xi^r \omega, \tag{4.7}$$

$$\left[\upsilon_r^i \mathcal{E}_i - d_\mu\left(\upsilon_r^{i,\mu} \mathcal{E}_i\right)\right] \xi^r \omega = 0. \tag{4.8}$$

Any Lagrangian L has gauge symmetries. In particular, there always exist trivial gauge symmetries

$$\upsilon = \sum_\Lambda \eta(M)_r^{i,\Lambda} \xi_\Lambda^r \partial_i, \qquad M_r^{i,\Lambda} = \sum_\Sigma T^{i,j,\Lambda,\Sigma} d_\Sigma \mathcal{E}_j, \qquad T_r^{j,i,\Lambda,\Sigma} = -T_r^{i,j,\Sigma,\Lambda},$$

corresponding to the trivial Noether identity

$$\sum_{\Sigma,\Lambda} T_r^{j,i,\Lambda,\Sigma} d_\Sigma \mathcal{E}_j d_\Lambda \mathcal{E}_i = 0.$$

Furthermore, given a gauge symmetry υ (3.11), let $E' \to Y$ be a vector bundle and h a linear differential operator on some composite bundle $E' \to Y \to X$, coordinated by (x^λ, y^i, ξ'^s), with the values in the vector bundle $E \to Y$. Then the composition

$$\upsilon' = \upsilon \circ h = \upsilon_s'^{i,\Lambda} \xi_\Lambda'^s \partial_i, \qquad \upsilon_s'^{i,\Lambda} = \sum_{\Xi+\Xi'=\Lambda} \sum_{0\leqslant|\Sigma|\leqslant m-|\Xi|} \upsilon_r^{i,\Xi+\Sigma} d_\Sigma h_s^{r,\Xi'},$$

is a variational symmetry of the pull-back of a Lagrangian L onto E', i.e., a gauge symmetry. In view of this ambiguity, we agree to say that a gauge symmetry υ (3.11) of a Lagrangian L is complete if any different gauge symmetry υ_0' of L factorizes through υ as

$$\upsilon' = \upsilon \circ h + T, \qquad T \approx 0.$$

A complete gauge symmetry always exists, but the vector bundle of its parameters need not be finite dimensional.

Accordingly, given the Noether operator (4.1), let H be a linear differential operator on $\overline{E}^* \to Y \to X$ with the values in the density-dual $\overline{E}'^*$ (1.1) of some vector bundle $E' \to Y$. Then the composition $\Delta' = H \circ \Delta$ is also a Noether operator. We agree to call the Noether operator (4.1) complete if a different Noether operator Δ' factors through Δ as

$$\Delta' = H \circ \Delta + F, \qquad F \approx 0.$$

Proposition 4.3. *A gauge symmetry υ of a Lagrangian L is complete iff the associated Noether operator is also.*

Proof. The proof follows at once from proposition 8.2 in the appendix. Given a gauge symmetry υ of L, let υ' be a different gauge symmetry. If $\eta(\upsilon)$ is a complete Noether operator, then

$$\eta(\upsilon') = H \circ \eta(\upsilon) + F, \qquad F \approx 0,$$

and, by virtue of relations (1.4), we have

$$\upsilon' = \upsilon \circ \eta(H) + \eta(F),$$

where $\eta(F) \approx 0$ because I_L is a differential ideal. The converse is similarly proved. □

5. Reducible gauge theories

Let us extend Noether's second theorem to the analysis of reducible gauge systems.

Definition 5.1. *A complete Noether operator $\Delta \not\approx 0$ (4.1) and the corresponding Noether identity (1.2) are said to be N-stage reducible ($N = 0, 1, \ldots$) if there exist vector bundles $E_k \to Y$ and differential operators Δ_k, $k = 0, \ldots, N$, such that:*

(i) Δ_k is a linear differential operator on the density-dual $\overline{E}^*_{k-1} \to Y \to X$ of $E_{k-1} \to Y$ with the values in the density-dual $\overline{E}^*_k$ of E_k, where $E_{-1} = E$;
(ii) $\Delta_k \not\approx 0$ for all $k = 0, \ldots, N$;
(iii) $\Delta_k \circ \Delta_{k-1} \approx 0$ for all $k = 0, \ldots, N$, where $\Delta_{-1} = \Delta$;
(iv) if Δ'_k is another differential operator possessing these properties, then it factors through Δ_k on-shell.

In particular, a zero-stage reducible Noether operator is called reducible. In this case, given bundle coordinates $(x^\lambda, y^i, \overline{\xi}_r)$ on $\overline{E}^*$ and $(x^\lambda, y^i, \xi^{r_0})$ on E_0, a differential operator Δ_0 reads

$$\Delta_0 = \sum_{0\leqslant|\Xi|\leqslant m_0} \Delta^{r,\Xi}_{r_0} \overline{\xi}_{\Xi r} \xi^{r_0} \omega. \tag{5.1}$$

Then the reduction condition $\Delta_0 \circ \Delta \approx 0$ takes the coordinate form

$$\sum_{0\leqslant|\Xi|\leqslant m_0} \Delta^{r,\Xi}_{r_0} d_\Xi \left(\sum_{0\leqslant|\Lambda|\leqslant m} \Delta^{i,\Lambda}_r \overline{y}_{\Lambda i} \right) \xi^{r_0} \omega \approx 0, \tag{5.2}$$

i.e., the left-hand side of this expression takes the form

$$\sum_{0\leqslant|\Sigma|\leqslant m_0+m} M^{i,\Sigma}_{r_0} \overline{y}_{\Sigma i} \xi^{r_0} \omega,$$

where all the coefficients $M^{i,\Sigma}_{r_0}$ belong to the ideal I_L.

Definition 5.2. *A complete gauge symmetry $\upsilon \not\approx 0$ (3.11) is said to be N-stage reducible if there exist vector bundles $E_k \to Y$ and differential operators υ^k, $k = 0, \ldots, N$, such that:*

(i) υ^k is a linear differential operator on the composite bundle $E_k \to Y \to X$ with values in the vector bundle $E_{k-1} \to Y$;
(ii) $\upsilon^k \not\approx 0$ for all $k = 0, \ldots, N$;
(iii) $\upsilon^{k-1} \circ \upsilon^k \approx 0$ for all $k = 0, \ldots, N$, where υ^k, $k = -1$, stands for υ;
(iv) if υ'^k is another differential operator possessing these properties, then υ'^k factors through υ^k on-shell.

Theorem 5.3. *A gauge symmetry υ is N-stage reducible iff the associated Noether identity is also.*

Proof. The proof follows at once from theorem 8.1 and proposition 8.2 in the appendix. Let us put $\Delta_k = \eta(\upsilon^k)$, $k = 0, \ldots, N$. If $\upsilon^k \approx 0$, then $\eta(\upsilon^k) \approx 0$ because I_L is a differential ideal. By the same reason, if υ^{k-1} and υ^k obey the reduction condition $\upsilon^{k-1} \circ \upsilon^k \approx 0$, then

$$\eta(\upsilon^{k-1} \circ \upsilon^k) = \eta(\upsilon^k) \circ \eta(\upsilon^{k-1}) \approx 0.$$

The converse is justified in a similar way. The equivalence of the conditions in items (iv) of definitions 5.1 and 5.2 is proved similarly to that in proposition 4.3. □

Remark 5.1. Let a gauge symmetry υ (3.11) be reducible. Given bundle coordinates $(x^\lambda, y^i, \xi^{r_0})$ on E_0, the differential operator υ^0 reads

$$\upsilon^0 = \sum_{0\leqslant|\Lambda|\leqslant m_0} \upsilon^{r,\Lambda}_{r_0} \xi^{r_0}_\Lambda \partial_r,$$

and the reduction condition $\upsilon \circ \upsilon^0 \approx 0$ takes the form

$$\sum_{0\leqslant|\Xi|\leqslant m} \upsilon^{i,\Xi}_r d_\Xi \left(\sum_{0\leqslant|\Lambda|\leqslant m_0} \upsilon^{r,\Lambda}_{r_0} \xi^{r_0}_\Lambda \right) \partial_i \approx 0.$$

In particular, it follows that the Noether current J (3.15) vanishes on-shell if

$$\xi^r = \sum_{0\leqslant|\Lambda|\leqslant m_0} \upsilon^{r,\Lambda}_{r_0} \xi^{r_0}_\Lambda$$

and, consequently, its components $J^{\lambda,\Lambda}_r$ are not independent.

6. Example I

This example addresses the gauge model of principal connections on a principal bundle $P \to X$ with a structure Lie group G whose automorphisms need not be vertical. In a general setting, the gauge-natural prolongations of P and the associated natural-gauge bundles can be considered [6].

Principal connections on a principal bundle $P \to X$ are represented by sections of the quotient

$$C = J^1 P/G \to X, \tag{6.1}$$

called the bundle of principal connections. This is an affine bundle coordinated by $\left(x^\lambda, a^r_\lambda\right)$ such that, given a section A of $C \to X$, its components $A^r_\lambda = a^r_\lambda \circ A$ are coefficients of the familiar local connection form (i.e., gauge potentials). We consider the GDA $\mathcal{O}^*_\infty[C]$.

Infinitesimal generators of one-parameter groups of automorphisms of a principal bundle P are G-invariant projectable vector fields on $P \to X$. They are associated with sections of the vector bundle $T_G P = TP/G \to X$. This bundle is endowed with the coordinates $(x^\lambda, \tau^\lambda = \dot{x}^\lambda, \xi^r)$ with respect to the fibre bases $\{\partial_\lambda, e_r\}$ for $T_G P$, where $\{e_r\}$ is the basis for the right Lie algebra $\mathbf{g}$ of G such that $[e_p, e_q] = c^r_{pq} e_r$. If

$$u = u^\lambda \partial_\lambda + u^r e_r, \qquad v = v^\lambda \partial_\lambda + v^r e_r, \tag{6.2}$$

are sections of $T_G P \to X$, their bracket reads

$$[u, v] = (u^\mu \partial_\mu v^\lambda - v^\mu \partial_\mu u^\lambda)\partial_\lambda + \left(u^\lambda \partial_\lambda v^r - v^\lambda \partial_\lambda u^r + c^r_{pq} u^p v^q\right) e_r.$$

Any section u of the vector bundle $T_G P \to X$ yields the vector field

$$u_C = u^\lambda \partial_\lambda + \left(c^r_{pq} a^p_\lambda u^q + \partial_\lambda u^r - a^r_\mu \partial_\lambda u^\mu\right)\partial^\lambda_r \tag{6.3}$$

on the bundle of principal connections C (6.1) [9].

In order to describe a gauge symmetry in this gauge model, let us consider the bundle product

$$E = C \underset{X}{\times} T_G P, \tag{6.4}$$

coordinated by $\left(x^\lambda, a^r_\lambda, \tau^\lambda, \xi^r\right)$. It can be provided with the generalized vector field

$$\upsilon_E = \upsilon = \left(c^r_{pq} a^p_\lambda \xi^q + \xi^r_\lambda - a^r_\mu \tau^\mu_\lambda - \tau^\mu a^r_{\mu\lambda}\right)\partial^\lambda_r. \tag{6.5}$$

With a subbundle $V_G P = VP/G \to X$ of the vector bundle $T_G P$ coordinated by (x^λ, ξ^r), we have the exact sequence of vector bundles

$$0 \to V_G P \longrightarrow T_G P \to TX \to 0.$$

The pull-back of this exact sequence via C admits the canonical splitting which takes the coordinate form

$$\tau^\lambda \partial_\lambda + \xi^r e_r = \tau^\lambda \left(\partial_\lambda + a^r_\lambda e_r\right) + \left(\xi^r - \tau^\lambda a^r_\lambda\right) e_r. \tag{6.6}$$

Due to this splitting, the generalized vector field (6.5) is brought into the form

$$\upsilon = \left(c^r_{pq} a^p_\lambda \xi'^q + \xi'^r_\lambda + \tau^\mu \mathcal{F}^r_{\lambda\mu}\right)\partial^\lambda_r, \qquad \xi'^r = \xi^r - \tau^\lambda a^r_\lambda. \tag{6.7}$$

This generalized vector field is a gauge symmetry of the global Chern–Simons Lagrangian in gauge theory on a principal bundle with a structure semi-simple Lie group G over a three-dimensional base X. Given a section B of $C \to X$ (i.e., a background gauge potential), this Lagrangian reads

$$\begin{aligned} L = \Big[&\tfrac{1}{2} a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\alpha \left(\mathcal{F}^n_{\beta\gamma} - \tfrac{1}{3} c^n_{pq} a^p_\beta a^q_\gamma\right) - \tfrac{1}{2} a^G_{mn} \varepsilon^{\alpha\beta\gamma} B^m_\alpha \left(F(B)^n_{\beta\gamma} - \tfrac{1}{3} c^n_{pq} B^p_\beta B^q_\gamma\right) \\ &- d_\alpha\left(a^G_{mn} \varepsilon^{\alpha\beta\gamma} a^m_\beta B^n_\gamma\right)\Big] \mathrm{d}^3 x, \end{aligned} \tag{6.8}$$

$$F(B)^r_{\lambda\mu} = \partial_\lambda B^r_\mu - \partial_\mu B^r_\lambda + c^r_{pq} B^p_\lambda B^q_\mu, \qquad \mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu,$$

where a^G is the Killing form [5, 11]. Its first term is the well-known local Chern–Simons Lagrangian, the second one is a density on X, and the Lie derivative of the third term is d_H-exact due to relation (3.3). The corresponding Noether identities (4.8) read

$$c^r_{pq} a^p_\lambda \mathcal{E}^\lambda_r - d_\lambda\left(\mathcal{E}^\lambda_q\right) = 0, \tag{6.9}$$

$$- a^r_{\mu\lambda} \mathcal{E}^\lambda_r + d_\lambda\left(a^r_\mu \mathcal{E}^\lambda_r\right) = 0. \tag{6.10}$$

The first one is the well-known Noether identity corresponding to the vertical gauge symmetry

$$\upsilon = \left(c^r_{pq} a^p_\lambda \xi^q + \xi^r_\lambda\right)\partial^\lambda_r.$$

The second Noether identity (6.10) is brought into the form

$$-a^r_\mu \left[c^r_{pq} a^p_\lambda \mathcal{E}^\lambda_r - d_\lambda\left(\mathcal{E}^\lambda_q\right)\right] + \mathcal{F}^r_{\lambda\mu} \mathcal{E}^\lambda_r = 0,$$

i.e., it is equivalent to the Noether identity $\mathcal{F}^r_{\lambda\mu} \mathcal{E}^\lambda_r = 0$, which also comes from the splitting (6.6) of the generalized vector field υ. This Noether identity however is trivial since $\mathcal{F}^r_{\lambda\mu} = 0$ is the kernel of the Euler–Lagrange operator of the Chern–Simons Lagrangian (6.8).

In order to obtain a gauge symmetry of the Yang–Mills Lagrangian, one should complete the generalized vector field (6.5) with the term acting on a world metric.

Let LX be the fibre bundle of linear frames in the tangent bundle TX of X. It is a principal bundle with the structure group $GL(n, \mathbb{R})$, $n = \dim X$, which admits reductions to its maximal compact subgroup $O(n)$. Global sections of the quotient bundle $\Sigma = LX/O(n)$ are Riemannian metrics on X. If X obeys the well-known topological conditions, pseudo-Riemannian metrics on X are similarly described. Being an open subbundle of the tensor bundle $\vee^2 TX$, the fibre bundle Σ is provided with bundle coordinates $\sigma^{\mu\nu}$. It admits the canonical lift

$$u_\Sigma = u^\lambda \partial_\lambda + (\sigma^{\nu\beta} \partial_\nu u^\alpha + \sigma^{\alpha\nu} \partial_\nu u^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} \tag{6.11}$$

of any vector field $u = u^\lambda \partial_\lambda$ on X. We describe the gauge system of principal connections and a dynamic metric field on the bundle product

$$E = C \underset{X}{\times} \Sigma \underset{X}{\times} T_G P, \tag{6.12}$$

coordinated by $(x^\lambda, a^r_\lambda, \sigma^{\alpha\beta}, \tau^\lambda, \xi^r)$. It can be provided with the generalized vector field

$$\upsilon = \left(c^r_{pq}a^p_\lambda\xi^q + \xi^r_\lambda - a^r_\mu\tau^\mu_\lambda - \tau^\mu a^r_{\mu\lambda}\right)\partial^\lambda_r + \left(\sigma^{\nu\beta}\tau^\alpha_\nu + \sigma^{\alpha\nu}\tau^\beta_\nu - \tau^\lambda\sigma^{\alpha\beta}_\lambda\right)\frac{\partial}{\partial\sigma^{\alpha\beta}}. \tag{6.13}$$

This is a gauge symmetry of the sum of the Yang–Mills Lagrangian $L_{\rm YM}(\mathcal{F}^r_{\alpha\beta}, \sigma^{\mu\nu})$ and a Lagrangian of a metric field. The corresponding Noether identities read

$$c^r_{pq}a^p_\lambda\mathcal{E}^\lambda_r - d_\lambda(\mathcal{E}^\lambda_q) = 0, \qquad -a^r_{\mu\lambda}\mathcal{E}^\lambda_r + d_\lambda(a^r_\mu\mathcal{E}^\lambda_r) - \sigma^{\alpha\beta}_\mu\mathcal{E}_{\alpha\beta} - 2d_\nu(\sigma^{\nu\beta}\mathcal{E}_{\mu\beta}) = 0.$$

The first one is the Noether identity (6.9). Then the second identity is brought into the form

$$\mathcal{F}^r_{\lambda\mu}\mathcal{E}^\lambda_r - 2\nabla_\nu(\sigma^{\nu\beta}\mathcal{E}_{\mu\beta}) = 0,$$

where ∇_ν are covariant derivatives with respect to the Levi-Civita connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu), \qquad K_\lambda{}^\mu{}_\nu = -\tfrac{1}{2}\sigma^{\mu\beta}(\sigma_{\lambda\beta\nu} + \sigma_{\nu\beta\lambda} - \sigma_{\beta\lambda\nu}).$$

7. Example II

Let us consider gauge theory of skew symmetric tensor fields. These are exterior forms on a base manifold X of degree more than one. We need not specify a gauge model, but refer to the topological BF theory [4]. This is a theory of two exterior forms A and B of form degree $|A| = \dim X - |B| - 1$. Another example is a gauge theory of an exterior form A in the presence of a background metric on X whose Lagrangian is similar to that of an electromagnetic field.

A generic gauge system of skew symmetric tensor fields is defined on the fibre bundle

$$Y = \overset{p}{\underset{X}{\wedge}} T^*X \underset{X}{\oplus} \overset{q}{\underset{X}{\wedge}} T^*X, \tag{7.1}$$

coordinated by $(x^\lambda, A_{\mu_1\ldots\mu_p}, B_{\nu_1\ldots\nu_q})$. The corresponding GDA is $\mathcal{O}^*_\infty[Y]$. There are the canonical p- and q-forms

$$A = \frac{1}{p!}A_{\mu_1\cdots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \in \mathcal{O}^{0,p}_\infty[Y],$$
$$B = \frac{1}{q!}B_{\nu_1\cdots\nu_q}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \in \mathcal{O}^{0,q}_\infty[Y] \tag{7.2}$$

on Y. A Lagrangian of the above-mentioned topological BF theory reads

$$L_{\rm BF} = A\wedge d_H B, \qquad p + q = n - 1. \tag{7.3}$$

A gauge symmetry of a generic gauge system of skew symmetric tensor fields, e.g., of the Lagrangian (7.3), is the following. Let us consider the fibre bundle

$$E = Y \underset{X}{\times} \overset{p-1}{\underset{X}{\wedge}} T^*X \underset{X}{\times} \overset{q-1}{\underset{X}{\wedge}} T^*X, \tag{7.4}$$

where

$$\overset{p-1}{\wedge} T^*X \underset{X}{\times} \overset{q-1}{\wedge} T^*X$$

is the fibre bundle of gauge parameters with coordinates $(x^\lambda, \varepsilon_{\mu_1\ldots\mu_{p-1}}, \xi_{\nu_1\ldots\nu_{q-1}})$. Let

$$\varepsilon = \frac{1}{(p-1)!}\varepsilon_{\mu_1\ldots\mu_{p-1}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p-1}} \in \mathcal{O}^{0,p-1}_\infty[E],$$
$$\xi = \frac{1}{(q-1)!}\xi_{\nu_1\ldots\nu_{q-1}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{q-1}} \in \mathcal{O}^{0,q-1}_\infty[E]$$

be canonical exterior forms like (7.2) on E (7.4). The above-mentioned gauge symmetry is given by the generalized vector field

$$\upsilon = d_{\mu_1}\varepsilon_{\mu_2\ldots\mu_p}\frac{\partial}{\partial A_{\mu_1\ldots\mu_p}} + d_{\nu_1}\xi_{\nu_2\ldots\nu_q}\frac{\partial}{\partial B_{\nu_1\ldots\nu_q}}, \tag{7.5}$$

which acts on the exterior forms A and B (7.2) by the law

$$\mathbf{L}_{\vartheta_E} A = d_H\varepsilon, \qquad \mathbf{L}_{\vartheta_E} B = d_H\xi. \tag{7.6}$$

In accordance with formula (4.8), the corresponding Noether identity takes the form

$$-\varepsilon_{\mu_2\ldots\mu_p} d_{\mu_1}\mathcal{E}^{\mu_1\ldots\mu_p} = 0, \qquad -\xi_{\nu_2\ldots\nu_q} d_{\nu_1}\mathcal{E}^{\nu_1\ldots\nu_q} = 0. \tag{7.7}$$

For instance, the equalities

$$\begin{aligned}\mathbf{L}_{\vartheta_E} L_{\mathrm{BF}} &= \big(\mathbf{L}_{\vartheta_E} A\big)\wedge d_H B + A\wedge\big(\mathbf{L}_{\vartheta_E} d_H B\big) = d_H\varepsilon\wedge d_H B + A\wedge\big(\mathbf{L}_{\vartheta_E} d_H B\big)\\ &= d_H(\varepsilon\wedge d_H B)\end{aligned}$$

show that the generalized vector field (7.5) is a gauge symmetry of the Lagrangian L_{BF} (7.3). This Lagrangian provides the Euler–Lagrange equations

$$d_H A = 0, \qquad d_H B = 0,$$

and the Noether identity (7.7) is brought into the form

$$d_H d_H A \equiv 0, \qquad d_H d_H B \equiv 0.$$

It should be emphasized that the gauge symmetry (7.5) by no means exhausts all variational symmetries of the Lagrangian L_{BF}. Any generalized vector field υ on Y such that $\mathbf{L}_\vartheta A$ is d_H-exact and $\mathbf{L}_\vartheta B$ is d_H-closed is a variational symmetry of this Lagrangian.

The gauge symmetry υ (7.5) is reducible. Without loss of generality, let us put $q \geqslant p$. Then υ is q-stage reducible as follows. Let us consider vector bundles

$$\begin{aligned} E_k &= Y \underset{X}{\times} \overset{p-k-2}{\underset{X}{\wedge}} T^*X \underset{X}{\times} \overset{q-k-2}{\underset{X}{\wedge}} T^*X, \qquad 0 \leqslant k < p-2,\\ E_k &= Y \underset{X}{\times} \mathbb{R} \underset{X}{\times} \overset{q-p}{\underset{X}{\wedge}} T^*X, \qquad k = p-2,\\ E_k &= Y \underset{X}{\times} \overset{q-k-2}{\underset{X}{\wedge}} T^*X, \qquad k > p-2,\end{aligned} \tag{7.8}$$

over Y provided with fibre coordinates

$$\big(\varepsilon^k_{\mu_1\ldots\mu_{p-k-2}}, \xi^k_{\nu_1\ldots\nu_{q-k-2}}\big), \qquad \big(\alpha, \xi^{p-2}_{\nu_1\ldots\nu_{q-p}}\big), \qquad \big(\xi^k_{\nu_1\ldots\nu_{q-k-2}}\big),$$

respectively. Then the differential operators

$$\begin{aligned}\upsilon_0 &= d_{\mu_1}\varepsilon^0_{\mu_2\ldots\mu_{p-1}}\frac{\partial}{\partial\varepsilon_{\mu_1\ldots\mu_{p-1}}} + d_{\nu_1}\xi^0_{\nu_2\ldots\nu_{q-1}}\frac{\partial}{\partial\xi_{\nu_1\ldots\nu_{q-1}}},\\ \upsilon_k &= d_{\mu_1}\varepsilon^k_{\mu_2\ldots\mu_{p-k-1}}\frac{\partial}{\partial\varepsilon^{k-1}_{\mu_1\ldots\mu_{p-k-1}}} + d_{\nu_1}\xi^k_{\nu_2\ldots\nu_{q-k-1}}\frac{\partial}{\partial\xi^{k-1}_{\nu_1\ldots\nu_{q-k-1}}}, \qquad 0<k<p-2,\\ \upsilon_{p-2} &= d_\mu\alpha\frac{\partial}{\partial\varepsilon^{p-3}_\mu} + d_{\nu_1}\xi^{p-2}_{\nu_2\ldots\nu_{q-p+1}}\frac{\partial}{\partial\xi^{p-3}_{\nu_1\ldots\nu_{q-p+1}}},\\ \upsilon_k &= d_{\nu_1}\xi^k_{\nu_2\ldots\nu_{q-k-1}}\frac{\partial}{\partial\xi^{k-1}_{\nu_1\ldots\nu_{q-k-1}}}, \qquad k>p-2,\end{aligned}$$

satisfy the conditions of definition 5.2.

Accordingly, there is a family of associated k-stage Noether operators Δ_k. Let the density duals $\overline{E}^* \to Y$, $\overline{E}_k^* \to Y$ of the vector bundles $E \to Y$ (7.4), $E_k \to Y$ (7.8) be provided with fibre coordinates

$$(\overline{\varepsilon}^{\mu_1\ldots\mu_{p-1}}, \overline{\xi}^{\nu_1\ldots\nu_{q-1}}), \qquad (\overline{\varepsilon}_k^{\mu_1\ldots\mu_{p-k-2}}, \overline{\xi}_k^{\nu_1\ldots\nu_{q-k-2}}), \qquad (\overline{\alpha}, \overline{\xi}_{p-2}^{\nu_1\ldots\nu_{q-p}}), \qquad (\overline{\xi}_k^{\nu_1\ldots\nu_{q-k-2}}),$$

respectively. Then we have

$$\Delta_0 = -\left[\varepsilon^0_{\mu_2\ldots\mu_{p-1}} d_{\mu_1}\overline{\varepsilon}^{\mu_1\ldots\mu_{p-1}} + \xi^0_{\nu_2\ldots\nu_{q-1}} d_{\nu_1}\overline{\xi}^{\nu_1\ldots\nu_{q-1}}\right]\omega,$$
$$\Delta_k = -\left[\varepsilon^k_{\mu_2\ldots\mu_{p-k-1}} d_{\mu_1}\overline{\varepsilon}_{k-1}^{\mu_1\ldots\mu_{p-k-1}} + \xi^k_{\nu_2\ldots\nu_{q-k-1}} d_{\nu_1}\overline{\xi}_{k-1}^{\nu_1\ldots\nu_{q-k-1}}\right]\omega, \qquad 0 < k < p-2,$$
$$\Delta_{p-2} = -\left[\alpha d_\mu \overline{\varepsilon}^\mu_{p-3} + \xi^{p-2}_{\nu_2\ldots\nu_{q-p+1}} d_{\nu_1}\overline{\xi}_{p-3}^{\nu_1\ldots\nu_{q-p+1}}\right]\omega,$$
$$\Delta_k = -\xi^k_{\nu_2\ldots\nu_{q-k-1}} d_{\nu_1}\overline{\xi}_{k-1}^{\nu_1\ldots\nu_{q-k-1}}\omega, \qquad k > p-2.$$

8. Appendix. Differential operators on dual fibre bundles

Given a fibre bundle $Y \to X$, let $E \to Y$, $Q \to Y$ be vector bundles coordinated by (x^λ, y^i, ξ^r) and (x^λ, y^i, q^a), respectively. Let E^*, Q^* be their duals and $\overline{E}^*$, $\overline{Q}^*$ their density-duals (1.1) coordinated by (x^λ, y^i, ξ_r), (x^λ, y^i, q_a) and $(x^\lambda, y^i, \overline{\xi}_r)$, $(x^\lambda, y^i, \overline{q}_a)$, respectively. Let υ be a linear Q-valued differential operator on $E \to Y \to X$. It is represented by function (2.10):

$$\upsilon = \upsilon^a q_a = \sum_{0\leqslant|\Lambda|\leqslant m} \upsilon_r^{a,\Lambda}(x^\lambda, y^i_\Sigma)\xi^r_\Lambda q_a \in \mathcal{O}^0_\infty[E \underset{Y}{\times} Q^*], \qquad 0 \leqslant |\Sigma| \leqslant m. \tag{A.1}$$

Let Δ be an m-order linear differential operator on the density-dual $\overline{Q}^* \to Y \to X$ of $Q \to Y$ with the values in the density-dual $\overline{E}^*$ of E. It is represented by the function

$$\overline{\Delta} = \sum_{0\leqslant|\Lambda|\leqslant m} \Delta_r^{a,\Lambda}(x^\lambda, y^i_\Sigma)\overline{q}_{\Lambda a}\xi^r \in \mathcal{O}^0_\infty[(\overline{E}^*)^* \underset{Y}{\times} \overline{Q}^*], \qquad 0 \leqslant |\Sigma| \leqslant m. \tag{A.2}$$

Let $J\omega$ be a volume form on X such that $\overline{\xi}_r = J\xi_r$ and $\xi^r = J\overline{\xi}^r$. Then function (A.2) defines the density $\Delta = \overline{\Delta}J\omega$ which reads

$$\Delta = \Delta_r\xi^r\omega = \sum_{0\leqslant|\Lambda|\leqslant m} \Delta_r^{a,\Lambda}(x^\lambda, y^i_\Sigma)\overline{q}_{\Lambda a}\xi^r\omega \in \mathcal{O}^{0,n}_\infty[E \underset{Y}{\times} \overline{Q}^*], \qquad 0 \leqslant |\Sigma| \leqslant m. \tag{A.3}$$

Theorem 8.1. *Any linear Q-valued differential operator υ (A.1) on $E \to Y \to X$ yields the linear $\overline{E}^*$-valued differential operator*

$$\eta(\upsilon) = \sum_{0\leqslant|\Sigma|\leqslant m} (-1)^{|\Sigma|} d_\Sigma\left(\upsilon_r^{a,\Sigma}\overline{q}_a\right)\xi^r\omega = \sum_{0\leqslant|\Lambda|\leqslant m} \eta(\upsilon)_r^{a,\Lambda}\overline{q}_{\Lambda a}\xi^r\omega,$$
$$\eta(\upsilon)_r^{a,\Lambda} = \sum_{0\leqslant|\Sigma|\leqslant m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma\left(\upsilon_r^{a,\Sigma+\Lambda}\right), \tag{A.4}$$

on $\overline{Q}^* \to Y \to X$. Conversely, any linear $\overline{E}^*$-valued differential operator Δ (A.3) on $\overline{Q}^* \to Y \to X$ defines the linear Q-valued differential operator

$$\eta(\Delta) = \sum_{0\leqslant|\Sigma|\leqslant m} (-1)^{|\Sigma|} d_\Sigma\left(\Delta_r^{a,\Sigma}\xi^r\right)q_a = \sum_{0\leqslant|\Lambda|\leqslant m} \eta(\Delta)_r^{a,\Lambda}\xi^r_\Lambda q_a,$$
$$\eta(\Delta)_r^{a,\Lambda} = \sum_{0\leqslant|\Sigma|\leqslant m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma\left(\Delta_r^{a,\Sigma+\Lambda}\right), \tag{A.5}$$

on $E \to Y \to X$. Relations (1.3) hold.

Proof. One must show that the differential operators given by the local coordinate expressions (A.4) and (A.5) are globally defined. The function υ (A.1) yields the density

$$\overline{\upsilon} = \sum_{0\leqslant|\Lambda|\leqslant m} \upsilon_r^{a,\Lambda}\xi_\Lambda^r\overline{q}_a\omega \in \mathcal{O}_\infty^{0,n}[E \underset{Y}{\times} Q^*]. \tag{A.6}$$

Its Euler–Lagrange operator

$$\delta(\overline{\upsilon}) = \mathcal{E}_i\, \mathrm{d}y^i \wedge \omega + \mathcal{E}_r\, \mathrm{d}\xi^r \wedge \omega + \mathcal{E}^a\, \mathrm{d}\overline{q}_a \wedge \omega$$

takes its values in the fibre bundle

$$V^*(E \underset{Y}{\times} Q^*) \underset{E \underset{Y}{\times} Q^*}{\otimes} \overset{n}{\underset{X}{\wedge}} T^*X, \tag{A.7}$$

where $V^*(E \underset{Y}{\times} Q^*)$ is the vertical cotangent bundle of the fibre bundle $E \underset{Y}{\times} Q^* \to X$. There is its canonical projection

$$\alpha_E : V^*(E \underset{Y}{\times} Q^*) \to V^*E \to V_Y^*E, \tag{A.8}$$

onto the vertical cotangent bundle V_Y^*E of $E \to Y$. Then we obtain a differential operator $(\alpha_E \circ \delta)(\overline{\upsilon})$ on $E \underset{Y}{\times} Q^*$ with the values in the fibre bundle $V_Y^*E \underset{E}{\otimes} \overset{n}{\wedge} T^*X$. It reads

$$(\alpha_E \circ \delta)(\overline{\upsilon}) = \mathcal{E}_r\, \overline{\mathrm{d}}\xi^r \otimes \omega = \sum_{0\leqslant|\Lambda|\leqslant m} (-1)^{|\Lambda|} d_\Lambda\left(\upsilon_r^{a,\Lambda}\overline{q}_a\right) \overline{\mathrm{d}}\xi^r \otimes \omega,$$

where $\{\overline{\mathrm{d}}\xi^r\}$ is the fibre basis for $V_Y^*E \to E$ and the tensor product $\otimes$ is over $C^\infty(X)$. Due to the canonical isomorphism $V_Y^*E = E^* \underset{Y}{\times} E$, this operator defines density (A.4). Conversely, the Euler–Lagrange operator of density (A.3) takes its values in the fibre bundle (A.7) and reads

$$\delta(\Delta) = \mathcal{E}_i\, \mathrm{d}y^i \wedge \omega + \mathcal{E}_r\, \mathrm{d}\xi^r \wedge \omega + \mathcal{E}^a \mathrm{d}\overline{q}_a \wedge \omega. \tag{A.9}$$

In order to repeat the above-mentioned procedure, let us consider a volume form $J\omega$ on X and substitute $\mathrm{d}\overline{q}_a \wedge \omega = J\mathrm{d}q_a \wedge \omega$ into expression (A.9). Using the projection

$$\alpha_Q : V^*(E \underset{Y}{\times} Q^*) \to V_Y^*Q^*$$

like α_E (A.8) and the canonical isomorphism $V_Y^*Q^* = Q \underset{Y}{\times} Q^*$, we come to the density

$$\sum_{0\leqslant|\Lambda|\leqslant m} (-1)^{|\Lambda|} d_\Lambda\left(\Delta_r^{a,\Lambda}\xi^r\right) q_a J\omega \in \mathcal{O}_\infty^{0,n}[E \underset{Y}{\times} Q^*]$$

and, hence, function (A.5). Relations (1.3) result from relation (2.19). □

Relations (1.3) show that the intertwining operator η (A.4) and (A.5) provides a bijection between the sets $\mathrm{Diff}(E, Q)$ and $\mathrm{Diff}(\overline{Q}^*, \overline{E}^*)$ of differential operators (A.1) and (A.3).

Proposition 8.2. *Compositions of operators $\upsilon \circ \upsilon'$ and $\Delta' \circ \Delta$ obey relations (1.4).*

Proof. It suffices to prove the first relation. Let $\upsilon \circ \upsilon' \in \mathrm{Diff}(E', Q)$ be a composition of differential operators $\upsilon \in \mathrm{Diff}(E, Q)$ and $\upsilon' \in \mathrm{Diff}(E', E)$. Given fibred coordinates (ξ^r) on $E \to Y$, (ϵ^p) on $E' \to Y$ and $(\overline{q}_a)$ on $\overline{Q}^* \to Y$, this composition defines density (A.6)

$$\overline{\upsilon \circ \upsilon'} = \sum_\Lambda \upsilon_r^{a,\Lambda} d_\Lambda \left(\sum_\Sigma \upsilon_p'^{r,\Sigma} \epsilon_\Sigma^p \right) \overline{q}_a\omega.$$

Following relation (2.16), one can bring this density into the form

$$\sum_\Sigma \upsilon'^{r,\Sigma}_p \epsilon^p_\Sigma \sum_\Lambda (-1)^{|\Lambda|} d_\Lambda \left(\upsilon^{a,\Lambda}_r \overline{q}_a\right)\omega + d_H\sigma = \sum_\Sigma \upsilon'^{r,\Sigma}_p \epsilon^p_\Sigma \sum_\Lambda \eta(\upsilon)^{a,\Lambda}_r \overline{q}_{\Lambda a}\omega + d_H\sigma.$$

Its Euler–Lagrange operator projected to $V^*_Y E' \underset{E'}{\otimes} T^*X$ is

$$\sum_\Sigma (-1)^{|\Sigma|} d_\Sigma \left(\upsilon'^{r,\Sigma}_p \sum_\Lambda \eta(\upsilon)^{a,\Lambda}_r \overline{q}_{\Lambda a}\right) \overline{\mathrm{d}}\epsilon^p \otimes \omega$$
$$= \sum_\Sigma \eta(\upsilon')^{r,\Sigma}_p d_\Sigma \left(\sum_\Lambda \eta(\upsilon)^{a,\Lambda}_r \overline{q}_{\Lambda a}\right) \overline{\mathrm{d}}\epsilon^p \otimes \omega,$$

that leads to the desired composition $\eta(\upsilon') \circ \eta(\upsilon)$. □

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